Int. J. Heat Mass Transfer. Vol. 19, pp. 799–803. Pergamon Press 1976. Printed in Great Britain

MEASUREMENT OF WALL REGION TURBULENT PRANDTL NUMBERS IN SMALL TUBES

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(*Received* 27 *November* 1975)

Abstract—Recent numerical analytic methods for the thermal entry region are synthesized with measurements of the local Nusselt number to provide a diagnostic technique to deduce the turbulent Prandtl number profile in the wall region from measurements in the thermal entry. The technique provides a means to obtain information on the turbulent Prandtl number even when other considerations, such as cost or avoidance of buoyancy effects, require that test section dimensions be too small for direct measurements with probes. For air flow at $Re = 4.45 \times 10^4$ in a circular tube the data show

$$Pr_t(y) = Pr_{t,w} + Pr_t' \cdot \frac{y}{r_w} = 0.9 \pm (0.0) \cdot \frac{y}{r_w}$$

within 10% in the wall region; these results refute a number of popular hypotheses for Pr_t.

NOMENCLATURE

- c_p, specific heat at constant pressure;
- D, inside diameter of test section;
- D_v, mass diffusivity;
- G, mass flow rate per unit area;
- g_c, dimensional conversion factor;
- h, heat-transfer coefficient;
- k, thermal conductivity;
- q'', heat flux;
- r, radius;
- T, temperature;
- u, velocity in axial direction;
- x, axial distance from start of heating;
- y, radial distance from wall.

Greek symbols

- α, thermal diffusivity, k/c_p;
- ε_h, eddy diffusivity for heat;
- ε_m, eddy diffusivity for momentum;
- κ, empirical constant in the van Driest mixing length, 0.4;
- μ, absolute viscosity;
- ν, kinematic viscosity;
- ρ, density;
- τ, shear stress.

Non-dimensional parameters

- f, friction factor, $2g_c\rho\tau_w/G^2$;
- Nu, Nusselt number, hD/k;
- Pr, Prandtl number, $c_p\mu/k$;
- q^+, heat flux parameter, q''_w/Gc_pT_{in};
- Re, Reynolds number, GD/μ;
- Sc, Schmidt number, $\mu/\rho D_v$;
- y^+, wall distance parameter, $y(g_c\tau_w/\rho)^{\frac{1}{2}}/\nu$;
- y_l^+, empirical constant in van Driest mixing length model, 26.

Subscripts

- b, bulk;
- DKM, Drew, Koo and McAdams;
- in, inlet;
- t, turbulent;
- vD, van Driest;
- w, wall.

1. INTRODUCTION

FOR PREDICTIONS of heat transfer to turbulent flows a key to the success of many engineering analyses is the appropriate choice of turbulent Prandtl number, $Pr_t = \varepsilon_m/\varepsilon_h$. Once the mean velocity field has been predicted satisfactorily, the turbulence model may then be extended directly to provide the mean temperature distribution and heat-transfer coefficients by well known analytical techniques.

Reynolds [1] recently presented an extensive review of studies of the turbulent Prandtl number for conditions approaching the constant fluid properties idealization without buoyant effects. The reader is referred to this review for further background. For the most part he dismisses experimental evidence as contradictory and only "a very coarse filter" for hypothesized models; this situation is partly the fault of the experimentalists. Very few have conducted a critical analysis of the experimental uncertainties in their results [2]. Moffat and co-workers [3] are among the few who present the uncertainty with the result so that the analyst can see where the data do in fact verify or refute his predictions; they show that the uncertainty envelope increases as the wall is approached. Small systematic errors become increasingly important. As the present study shows, for heating applications predictions of the Nusselt number and wall temperature in some conditions are most sensitive to the value of Pr_t near the wall.

The purpose of the present paper is to introduce an alternative method of deducing Pr_t in the region near the wall and to demonstrate its application. Briefly, the idea is to use numerical predictions to relate the distribution of $Pr_t(y)$ to the axial variation of the local Nusselt number in the immediate thermal entry of a tube flow with a fully developed, entering velocity profile. Experimental measurements of $Nu(x)$ may then be used to infer Pr_t for the regions in which the relationship is most sensitive and, thereby, most important.

While several authors, such as Na and Habib [4], have used computer experiments with a particular turbulence model to force proper prediction of the Nusselt number in the fully developed region, it appears that none has concentrated on the immediate thermal entry where the boundary layer growth is dominated by $Pr_{t,w}$. Kays [5] reports a computer experiment matching measured temperature profiles away from the wall to deduce $Pr_t(y)$ but, like the direct measurements of Simpson *et al.* [3], the thermal boundary layer was already well developed at the location considered. The new contribution of the proposed technique is the extension of wall parameter methods into the immediate thermal entry to deduce $Pr_{t,w}$. As a consequence, information about $Pr_t(y)$ can be obtained for situations where direct methods using internal probes are impossible, e.g. use of small diameter test sections to reduce expense or to avoid buoyancy effects. The application which initiated the study involves heat transfer to the expensive noble gases.

2. BACKGROUND ANALYSIS

In this report, analysis is by the numerical method of Bankston and McEligot [6] utilizing finite control volume approximations to solve the energy equation,

$$u\frac{\partial t}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left\{(\alpha + \varepsilon_m/Pr_t)\frac{\partial t}{\partial r}\right\}. \quad (1)$$

Idealizations implied by this form are (a) the axisymmetric boundary layer approximations, (b) fully developed, steady flow at low velocities, and (c) constant fluid properties. Solution with the measured inlet conditions and with the measured wall heat flux variation as a boundary condition leads to prediction of the Nusselt number.

Examination of the energy equation (1) shows that, with the fluid properties and boundary conditions known, the functional dependence of the result can be described as

$$Nu = Nu\{x, u(r), \varepsilon_m(r), Pr_t\} \quad (2a)$$

or, since $\varepsilon_m(r)$ determines $u(r)$ for fully developed flow,

$$Nu = Nu\{x, \varepsilon_m(r), Pr_t\}. \quad (2b)$$

For adiabatic flow in circular tubes, several semi-empirical descriptions of $\varepsilon_m(r)$ are now generally well accepted; those to van Driest [7] and Reichardt [8] are not seriously disputed for practical applications. Thus, $Nu = Nu(x, Pr_t)$. Conceptually, the relationship could be inverted to $Pr_t = Pr_t\{Nu(x)\}$ provided that Pr_t can be considered as one-dimensional (with the flow fully developed, it is reasonable to assume that $Pr_t(x, r) = Pr_t(r)$). Blom [9] has shown with his own measurements and others that this assumption is valid near the wall. Careful experiments measuring $Nu(x)$ can then be used to deduce $Pr_t(r)$. Unfortunately, it does not appear that direct inversion can be done easily, if at all, so the proposed method employs iterative use of an existing numerical program.

Sensitivity of the proposed method is examined for a typical experiment with gas flow through a vertical circular tube with electrical resistive heating which provides an approximately constant heat flux. The simple van Driest mixing length model [7] is used to calculate $\varepsilon_m(r)$ and $u(r)$ in this examination. Grid parameters of the numerical program are chosen to yield solutions within about 1% of the converged values.

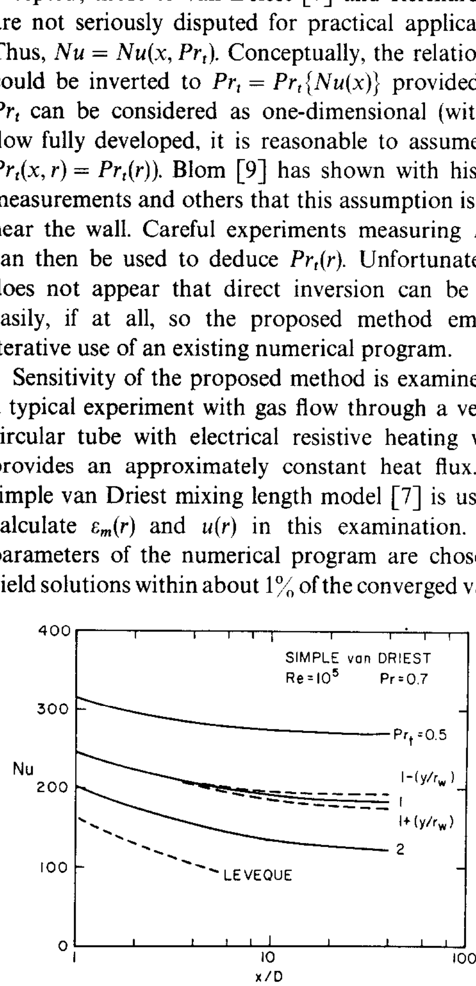

FIG. 1. Effect of $Pr_t(y)$ distribution for gas flow in thermal entry.

The results, Fig. 1, demonstrate the relative sensitivity of $Nu(x)$ to variation of Pr_t near the wall and in the central flow. At the hypothesized condition ($Re = 10^5$ and $Pr = 0.7$), a constant value of Pr_t leads to significant differences in $Nu(x)$ depending on the value chosen. At $x/D = 1$ a change from $Pr_t = 1$ to $Pr_t = 1/2$ leads to a 30% change in Nu, and at $x/D = 40$ it leads to 45%. To examine which region of the flow dominates, three Pr_t profiles are compared: $Pr_t = 1$, $1 + (y/r_w)$, and $1 - (y/r_w)$. The second corresponds to the first at the wall and to $Pr_t = 2$ at the centerline while the third varied from 1 to 0 giving an average of 1/2; thus, these Pr_t profiles are equivalent to each other at the wall and differ substantially in the core. At $x/D = 1$, differences between the results are insignificant; then they diverge downstream. At these conditions the Nusselt number measured at small values of x/D would determine $Pr_{t,w}$, and its variations as x/D increases would indicate the radial variation of $Pr_t(y)$.

The relative sensitivity can be explained by considering the growth of the thermal boundary layer starting at $x = 0$. Initially, the solution corresponds to the Leveque solution [10]; this situation continues until the thermal boundary layer grows beyond the linear sublayer ($u^+ \approx y^+$) or until ε_h is no longer much smaller than α. Then the growth, or thermal entrainment, is determined by Pr_t in the viscous layer ($y^+ \gtrsim 30$) and as the flow proceeds downstream the turbulence properties at successively greater distance from the wall become important progressively. Consequently, at small values of x the local Nusselt number depends only on Pr_t near the wall, while further downstream $Nu(x)$ is affected by Pr at larger y. The experimental uncertainty in $Pr_t(y)$ can be determined from the experimental uncertainty in $Nu(x)$ or $t_w(x)$. For the conditions of Fig. 1 we see that an experimental uncertainty of 5% in Nu near $x/D = 1$ would lead to about 8% uncertainty in $Pr_{t,w}$.

Experimental precautions necessary to obtain meaningful measurements of $Pr_t(y)$ by this technique are discussed elsewhere [11].

3. DEMONSTRATION OF TECHNIQUE

The experiment chosen for demonstration was performed on apparatus essentially the same as Campbell and Perkins [12] except a circular tube was employed. Air served as the gas. The test section was an 1/8th in (3.2 mm) Hastelloy-X tube with a 92 diameter unheated entry and 98 diameter heated length. Wall thickness was 0.022 in (0.6 mm).

The lowest temperature run in a series investigating effects of property variation is investigated. At $Re_{in} = 4.45 \times 10^4$ the tube was heated at a rate $q^+ = (q''_w/Gc_p T_{in}) = 0.00057$, which gives a maximum temperature ratio (T_w/T_b) of 1.185. A preliminary adiabatic run agreed with the Drew, Koo and McAdams correlation within 1.3%. Exit Mach number was 0.076 and the Grashof/(Reynolds)^2 quotient was less than 0.0002. The axial heat flux distribution resembled a short exponential approach to a slightly decreasing ramp function due to the heat loss distribution. Axial conduction loss at the first thermocouple was about 18% of the energy generation rate while the maximum radial heat loss was about 5% (energy generation varied by only 1.4% axially due to the increase in electrical resistivity with temperature). For such experiments the experimental uncertainties in Nusselt number are typically of the order of 10% at the entrance and about 5% away from the electrodes.

The Nusselt numbers for constant property conditions were deduced by multiplying the measured values by the local value of $(T_w/T_b)^{0.5}$ as suggested by Kays [13] for the effects of gas property variation. These data are presented on Fig. 2 along with brackets showing representative uncertainty estimates. Wall temperature measurements were not available upstream to implement the calculation of a premature start of the thermal boundary (due to axial conduction in the tube wall) in the numerical analysis as by Shumway and

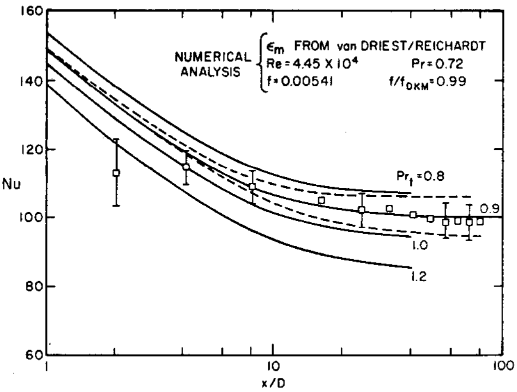

FIG. 2. Determination of $Pr_t(y)$ in wall region from measured $Nu(x)$.

McEligot [14]. However, numerical predictions varying the axial heat flux distribution showed that the difficulties in the immediate thermal entry become negligible by about eight to ten diameters; a thinner tube would reduce this distance.

As noted in Section 2, the determination of Pr_t by the proposed technique depends on use of a proper distribution for $\varepsilon_m(r)$. In addition to providing an acceptable velocity profile, $\varepsilon_m(r)$ should lead to a value of the friction factor which agrees with accepted correlations for adiabatic friction factors. For demonstration of the technique we use a combination of the van Driest mixing length [7] and the Reichardt middle law [8]

$$l_{vD} = \kappa y\{1 - \exp(-y^+/y_l^+)\} \tag{3a}$$

and

$$\varepsilon_m = \varepsilon_{vD} \cdot \left(2 - \frac{y}{r_w}\right) \cdot \left[1 + 2\left(\frac{r}{r_w}\right)^2\right] \Big/ 6. \tag{3b}$$

With $\kappa = 0.4$ and $y_l^+ = 26$, this representation yields friction factors within about 1% of the Drew, Koo and McAdams correlation [15] over the range $3 \times 10^4 < Re < 3 \times 10^5$.

The turbulent Prandtl number in the vicinity of the wall was deduced manually for the present paper, but the following procedure could be programmed for direct iteration by the numerical analysis program or for interactive control via computer graphics. Ignoring the first two data points, we concentrated on estimating $Pr_{t,w}$ from the data near $x/D = 10$ by plotting results based on trial constant values of Pr_t(0.8, 1.0 and 1.2) as shown in Fig. 2. The value $Pr_{t,w} = 0.9$ appeared reasonable as a first approximation. Then the variation with distance from the wall was deduced from trials of the form

$$Pr_t = Pr_{t,w} + \frac{dPr_t}{d(y/r_w)} \cdot \left(\frac{y}{r_w}\right) \tag{4}$$

applied to the data further downstream. Trials with $Pr_t = 0.9 + 0.9(y/r_w)$ (dashed), 0.9 (solid) and $0.9 - 0.9(y/r_w)$ (dashed) are shown plotted. For these data the deduced experimental value is seen to be

$$Pr_{t,w} = 0.9 \pm 0.1$$

and

$$dPr_t/d(y/r_w) = 0$$

for the range of importance. A next iteration might start from $Pr \approx 0.88+0.02(y/r_w)$ but the precision of these data does not warrant extension further.

4. DISCUSSION

Comparison of Figs. 1 and 2 shows no great change in the sensitivity, $\partial Pr_t/\partial Nu$, as a function of Reynolds number but its range is only a factor of about two. One would expect that the sensitivity would vary with Re because the wall layer occupies a greater fraction of the tube cross section as Re is reduced. Likewise, variation of the molecular Prandtl number would be expected to have an effect. Examination of these aspects is an appropriate task for further study as the technique is employed, but is not considered in the scope of the present work.

Since a thermal entry precedes fully established conditions, the proposed technique can be used whenever $Pr_t(y)$ is determined by the usual method of measuring $t(y)$ and $u(y)$ well downstream. Then the wall measurements can be applied either to extend or to test such "direct" Pr_t profile measurements. The numerical programs required are now readily available or easily developed [6, 16]. The wide disparity of Pr_t measurements [1] indicates that such testing is probably necessary.

One of the main advantages of the present method is that it restricts concentration to the region which causes significant effects on heat transfer. The relative insensitivity of $Nu(x)$ to variation of Pr_t near the centerline implies that determination of Pr_t in that region is not important to the engineer attempting to predict surface temperatures in flows heated from the wall for the sort of conditions considered herein.

While one experimental run is not sufficient to establish the functional dependence of $Pr_t(y)$ on such parameters as Re and Pr, it is enough to refute or to provide partial evidence in support of available hypotheses. In Fig. 3 the present data are superimposed on a presentation of existing hypotheses and data provided by Quarmby and Quirk [17]. It is readily seen that most shown do not have a reasonable asymptote as the wall is approached. On the other hand, the present result is in agreement with the data by Meier and Rotta [18] for turbulent boundary layers.

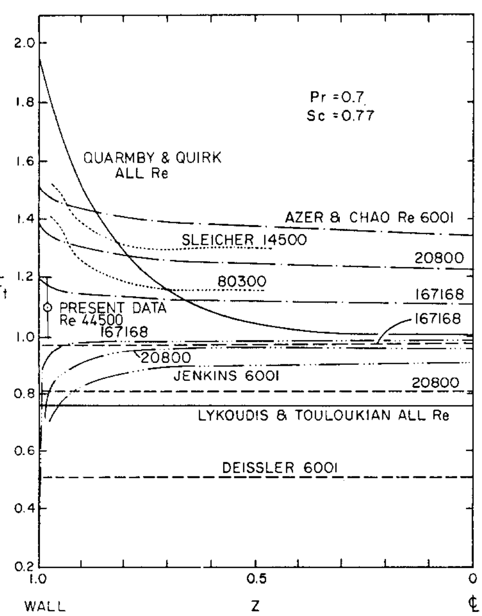

FIG. 3. Comparison of present data to variation of $Pr_t(y)$ for gas flows as suggested by various authors [17].

5. CONCLUSIONS

By careful measurements of wall parameters in the thermal entry of small tubes, it is possible to determine $Pr_t(y)$ for the range of y/r_w for which knowledge of Pr_t is most important in wall heating applications.

For air flow at $Re = 4.45 \times 10^4$

$$Pr_t(y) = Pr_{t,w} + Pr_t' \cdot \frac{y}{r_w} = 0.9 + (0.0) \cdot \frac{y}{r_w}$$

within 10% in the wall region.

Acknowledgements—This work forms a part of studies supported by the Office of Naval Research (U.S.A.) and by the Bundesministerium für Forschung und Technologie (BRD).

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MESURE DU NOMBRE DE PRANDTL TURBULENT PRES DE LA PAROI DANS DES PETITS TUBES

Résumé—Les méthodes récentes d'analyse numérique appliquées à la région d'établissement du régime thermique utilisées en relation avec les mesures de nombres de Nusselt locaux ont fourni une technique de détermination du profil du nombre de Prandtl turbulent près de la paroi à partir de mesures dans la région d'établissement thermique. Cette technique fournit un moyen d'information sur la valeur du nombre de Prandtl turbulent même lorsque d'autres considérations, telles que le coût ou la réduction des effets de convection naturelle, nécessitent l'utilisation de sections d'étude trop petites pour des mesures directes à l'aide de sondes. Pour un écoulement d'air à $Re = 4{,}45 \cdot 10^4$ dans un tube circulaire, les données expérimentales fournissent;

$$Pr_t(y) = Pr_{t\,\mathrm{paroi}} + Pr_t' \cdot \frac{y}{r_w} = 0{,}9 \pm (0,0) \cdot \frac{y}{r_w}$$

à mieux de 10 pour cent près dans la région de paroi; ces résultats réfutent certaines hypothèses courantes sur Pr_t.

MESSUNGEN DER PRANDTL-ZAHL IM TURBULENTEN WANDNAHEN BEREICH KLEINER ROHRE

Zusammenfassung—Numerische analytische Methoden für den thermischen Einlaufbereich werden mit Messungen der örtlichen Nusselt-Zahl verglichen, um die Profile der Prandtl-Zahl bei turbulenten wandnahen Strömungen aus den Messungen im thermischen Einlaufbereich zu erhalten. Das Verfahren erlaubt eine Aussage über den turbulenten Bereich der Prandtl-Zahl selbst wenn die Berücksichtigung von Kosten oder die Verhinderung von Auftriebseffekten, die Dimensionen der Durchmesser der Versuchsrohre als zu klein für direkte Messungen vorschreiben. Für Luftströmungen bei $Re = 4{,}45 \times 10^4$ in einem Kreisrohr folgen die Ergebnisse folgender Beziehung

$$Pr_t(y) = Pr_{t,\mathrm{wand}} + Pr_t' \frac{y}{r_w} = [0{,}9 \pm (0{,}0)] \frac{y}{r_w}$$

mit einer Streuung von 10%. Diese Ergebnisse widerlegen eine Reihe von bekannten Hypothesen für Pr_t.

ИЗМЕРЕНИЕ ТУРБУЛЕНТНЫХ ЗНАЧЕНИЙ ЧИСЕЛ ПРАНДТЛЯ В ПРИСТЕННОЙ ОБЛАСТИ В НЕБОЛЬШИХ ТРУБАХ

Аннотация — На основании последних численных аналитических методов расчёта теплового начального участка и измерений локального числа Нуссельта создана диагностическая методика расчёта турбулентного числа Прандтля в пристенной области по измерениям для начального теплового участка. Эта методика даёт информацию о турбулентном числе Прандтля даже тогда, когда в силу других соображений, таких как использование или пренебрежение эффектами Архимедовой силы, требуются слишком малые размеры экспериментального участка для непосредственных измерений с помощью датчиков. Для воздушного потока при $Re = 4.45 \times 10^4$ в трубе круглого сечения получили:

$$Pr_t(y) = Pr_{t,\,\mathrm{wall}} + Pr_t' \cdot \frac{y}{r_w} = 0.9 \pm (0.0) \cdot \frac{y}{r_w}$$

в пределах 10% для пристенной области; эти результаты опровергают целый ряд существующих гипотез относительно Pr_t.